Common maths for physicists

associativity $x\cdot(y\cdot z) = x\cdot y\cdot z = (x\cdot y)\cdot z$

commutativity $x\cdot y = y\cdot x$

distributivity $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$

reflexive $x = x$

symmetric $x = y \Rightarrow y = x$

transitive $x = y \wedge y = z \Rightarrow x = z$

identity (neutral) element of binary operation

$e \in X,\ x \times e = e \times x = x$

inverse element $x \times y = y \times x = e$

$$\sin^2\alpha = \tfrac{1}{2}[1-\cos(2\alpha)]$$
$$\cos^2\alpha = \tfrac{1}{2}[1+\cos(2\alpha)]$$
$$\sin^3\alpha = \tfrac{1}{4}[3\sin\alpha-\sin(3\alpha)]$$
$$\cos^3\alpha = \tfrac{1}{4}[3\cos\alpha+\cos(3\alpha)]$$

$$\cos(x) = \cos(-x)$$
$$\sin(x) = -\sin(-x)$$

$$\sin(x\pm y) = \sin x\ \cos y \pm \sin y\ \cos x$$
$$\cos(x\pm y) = \cos x\ \cos y \mp \sin x\ \sin y$$

$$\sin(2x) = 2\sin x\ \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$

$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x} \qquad \left|\sin\frac{x}{2}\right| = \sqrt{\frac{1-\cos x}{2}}$$
$$\cot(2x) = \frac{\cot^2 x - 1}{2\cot x} \qquad \left|\cos\frac{x}{2}\right| = \sqrt{\frac{1+\cos x}{2}}$$

$$\sin x \pm \sin y = 2\sin\left(\frac{x\pm y}{2}\right)\cdot\cos\left(\frac{x\mp y}{2}\right)$$
$$\cos x + \cos y = +2\cos\left(\frac{x+y}{2}\right)\cdot\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\cdot\sin\left(\frac{x-y}{2}\right)$$

$$2\cos x\ \cos y = \cos(x-y)+\cos(x+y)$$
$$2\sin x\ \sin y = \cos(x-y)-\cos(x+y)$$
$$2\sin x\ \cos y = \sin(x-y)+\sin(x+y)$$

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh(2x) = 2\sinh x\ \cosh x$$
$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

$$\sin\frac{\pi}{6} = \frac{1}{2}\ ,\quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\ ,\quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Complex numbers

$z = r\cdot e^{i\varphi};\ \ r = |z|,\ \ \varphi = Arg(z)$

$e^{ix} = \cos x + i\sin x$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2},\quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\sqrt[n]{r e^{i\varphi}} = \sqrt[n]{r}\, e^{i\left(\frac{\varphi+2k\pi}{n}\right)},\quad k = 0,1,\dots,n-1$$

$\mathrm{Ln}(z) = \ln|z| + i\{Arg(z)+2k\pi\};\ \ k\in\mathbb{Z}$

$$\arccos(z) = \frac{1}{i}\mathrm{Ln}\left(z+\sqrt{z_2-1}\right)$$

$$\arctan(z) = \frac{1}{2i}\mathrm{Ln}\frac{1+iz}{1-iz}$$

$$\mathrm{arcosh}(z) = \mathrm{Ln}\left(z+\sqrt{z^2-1}\right),\quad \mathrm{arsinh}(z) = \mathrm{Ln}\left(z+\sqrt{z^2+1}\right)$$

$$\mathrm{artanh}(z) = \frac{1}{2}\mathrm{Ln}\frac{1+z}{1-z},\quad \mathrm{arcoth}(z) = \frac{1}{2}\mathrm{Ln}\frac{1+z}{z-1}$$

Derivatives

$$(a^x)' = a^x\cdot\ln a$$
$$(\log_a x)' = \frac{1}{x\cdot\ln a}$$

$$(\sin x)' = \cos x$$
$$(\cos x)' = -\sin x$$
$$(\tan x)' = \frac{1}{\cos^2 x}$$
$$(\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$
$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$
$$(\arctan x)' = \frac{1}{1+x^2}$$
$$(\mathrm{arccot}\, x)' = -\frac{1}{1+x^2}$$

$$(\sinh x)' = \cosh x$$
$$(\cosh x)' = \sinh x$$

$$f^{-1}(y_0)' = \frac{1}{f(x_0)'}$$

$$\frac{d\,F(y)}{dy} = \frac{d}{dy}\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx =$$
$$\int_{\alpha(y)}^{\beta(y)} \frac{\partial f(x,y)}{\partial y}\,dx + f(\beta(y),y)\cdot\beta'(y)$$

Integrals

$$\int a^x dx = \frac{a^x}{\ln(a)}+C$$

$$\int \frac{dx}{\sqrt{x^2\pm 1}} = \ln\left|x+\sqrt{x^2\pm 1}\right|+C$$

$\int u v' dx = u v - \int v u' dx$ solution per partes (the same for definite integral, just add boundaries)

$$\int_0^{+\infty} x^{2k}\, e^{-ax^2} dx = \frac{1\cdot 2\cdot 3\cdot\ldots\cdot(2k-1)\sqrt{\pi}}{2^{k+1}a^{(2k+1)/2}}$$

$$\int_0^{+\infty} x^{2k+1}\, e^{-ax^2} dx = \frac{k!}{2a^{k+1}}$$

$\int_0^{+\infty} e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a}$ Laplace-Gauss integral

$$\int_0^{\frac{\pi}{2}} \sin^{\alpha}(x)\cos^{\beta}(x)\,dx = \frac{1}{2}B\left(\frac{\alpha+1}{2},\frac{\beta+1}{2}\right)$$

$\int_{-\infty}^{\infty}\cos(x^2) = \int_{-\infty}^{\infty}\sin(x^2) = \sqrt{\frac{\pi}{2}}$ Fresnel interals

$$F(\beta) = \int_0^\infty \frac{\sin \beta x}{x} dx = \begin{cases} \frac{\pi}{2} & \beta > 0 \\ 0 & \beta = 0 \\ -\frac{\pi}{2} & \beta < 0 \end{cases}$$ Dirichlet integral

$$\iiint_V \operatorname{div} \vec{E}\, dV = \iint_{\partial V} \vec{E} \cdot \vec{dS} \;\rightarrow\; \vec{dS} = \vec{N} \cdot dS$$ Gauss–Ostrograd

$$\oint_{L=\partial S} \vec{E} \cdot d\vec{r} = \iint_S \operatorname{rot} \vec{E} \cdot d\vec{S}$$ Stokes formula

Cauchy–Riemann equations

$w = f(z) = f(x + iy) = u(x, y) + i v(x, y)$
in complex plane is existence of derivative in some point equivalent to being continuous in that point for function

$$\exists f'(z_0) \Rightarrow \begin{matrix} \partial_x u = \partial_y v \\ \partial_x v = -\partial_y u \end{matrix}$$ Cauchy–Riemann conditions

$$f'(z_0) = \partial_x u + i \partial_x v \Big|_{z_0} = \partial_y v - i \partial_y u \Big|_{z_0}$$

Real and Imaginary part of Analytic function (has continuous first derivative in open continuous area) are harmonic functions.

Function $U(x, y)$ is harmonic in D when it is continuous function in $\bar{D} = D \cup \partial D$, has continuous second derivatives and meets Laplace partial diferential equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \Delta U = 0 ;\; (x, y) \in D$$

Cauchy integral theorem

In the following Cauchy integral ... is orientation of contour the same and counter-clockwise (↺).

for $f(z)$ analytic on $D \cup (C = \partial D)$ is

$$\oint_C f(z) dz = 0 \quad \left(\oint_C f(z) dz = \sum_{k=1}^{n} \oint_{C_k} f(z) dz \right)$$

Cauchy integral formula

for $f(z)$ analytic on $D \cup (C = \partial D)$, $C \cap D_C = \emptyset$ is

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{\zeta - z} \quad \forall z \in D_C \text{ (area without holes)}$$

$$\left(f(z) = \frac{1}{2\pi i} \left[\oint_C \frac{f(\zeta) d\zeta}{\zeta - z} - \sum_{k=1}^{n} \oint_{C_k} \frac{f(\zeta) d\zeta}{\zeta - z} \right] \right)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} ;\; n = 0, 1, \ldots$$

Residue

z is isolated singularity of analytic function $f(z)$ than

$$\underset{\text{Laurent series}}{A}_{-1} = \operatorname*{Res}_{z=a} \{f(z)\} = \frac{1}{2\pi i} \oint_{\Gamma : |z-a| = r ;\; 0 < r < \varepsilon}^{\Gamma} f(\zeta) d\zeta$$

is residue. It is used for:

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} \{f(z)\}$$ where a_k are isolated singularities. $f(z)$ is analytic on $D \cup (C = \partial D)$ except final number of isolated singularities a_k and where C is simple contour and $a_k \in D_C ;\; k = 1, \ldots, n ;\; D_C \cap D = \emptyset$

Common definitions

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \qquad e^{-1} = \lim_{n \to \infty} \left(1 + \frac{-1}{n}\right)^n$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Special functions

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt ,\; \forall x \in (0, \infty)$$ Gamma function

$$\Gamma(x+1) = x\,\Gamma(x) \quad x \in (0, \infty)$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad x \in (0, 1)$$

$$2^{2x-1} \Gamma(x) \Gamma\left(x + \frac{1}{2}\right) = \sqrt{\pi}\, \Gamma(2x) \quad x \in (0, 1)$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n-1)!!}{2^n} \sqrt{\pi} \quad n \in \mathbb{N}$$

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \quad x, y \in \mathbb{R}^+$$ Beta function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Series

Fourier series

$f(x) :\; 2l = T$

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx$$

$$\frac{1}{l} \int_{-l}^{l} f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$ Parseval identity or equation which proves that system of trigonometric functions is closed

Tailor formula

$f(x) = T_n(x, a) + R_n(x, a)$

$$T_n(x, a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$R_n(x,a) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$$

what is Lagrange form of remainder (residual) where

$c \in \langle a, x\rangle,\ \ c = a + \vartheta(x-a)$

Newton binomial series

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k}$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n} x^n = \sum_{n=0}^{\infty}\frac{\alpha!}{(\alpha-n)!\,n!}x^n$$

from Tailor for $\forall \alpha \in \mathbb{R},\ \ \forall x : |x| < 1$

Infinite series

Infinite sequence is convergent if the sequence of partial sums (S_n) converges. Convergence tests for infinite sequences:

Series with nonnegative elements

Cauchy integral criterion

$f(x) \geqslant 0,\ \ f \in C_{\langle 0,+\infty\rangle}$ (i.e. continuous function) and f is not increasing function on $\langle 0,+\infty)$

$$\rightarrow \int_0^{+\infty} f\,dx \text{ and } \sum_{n=1}^{\infty} f(n) \text{ are similar}$$

with $L_{\text{something}}() \equiv \lim\limits_{\text{appropriate variable}\rightarrow\text{something}}()$

D'Alombert criterion $\quad L_\infty \frac{a_{n+1}}{a_n} = q \begin{cases} <1 & \text{convergent} \\ >1 & \text{divergent}\end{cases}$

Cauchy criterion $\quad L_\infty \sqrt[n]{a_n} = \begin{cases} <1 & \text{convergent} \\ >1 & \text{divergent}\end{cases}$

Raab criterion $\quad L_\infty n\cdot\left(1-\frac{a_{n+1}}{a_n}\right) = b \begin{cases} <1 & \text{diver} \\ >1 & \text{conv}\end{cases}$

Series with alternating signs

Leibnitz: say b_n is series with alternating signs and $a_n \downarrow 0 = 0$ (i.e. is monotonically decreasing (increasing) to 0) than $b_n a_n$ is convergent

Abel: $\sum^\infty b_n$ is convergent and $\{a_n\}$ is monotonic and bounded than $\sum^\infty a_n b_n$ is convergent

Dirichlet: $S_N \sum^\infty b_n$ are bounded and $\{a_n\}\downarrow 0 = 0$ than $\sum^\infty a_n b_n$ is convergent

Fourier transformation

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt\right\}e^{-i\omega x}d\omega$$

and $\int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|F(\omega)|^2 d\omega$

Calculus of variations

Functional of n functions

$$J[y_1(x),\ldots,y_n(x)] = \int_a^b F[x, y_1(x), y_1'(x),\ldots,y_n(x),y_n'(x)]dx$$

$$+\text{boundary conditions}\quad y_k(a) = A_k,\ \ y_k(b) = B_k\ \Big|_{k=1,\ldots,n}$$

Solution vector have to be solution of Euler's equations:

$$\frac{d}{dx}\left[\frac{\partial F}{\partial y_k'}\right] - \frac{\partial F}{\partial y_k} = 0\ \Big|_{k=1,\ldots,n}$$

Functional of two independent variables

$$J[u(x,y)] = \iint_{\bar{D}} F[x, y, u(x,y), \partial_x u, \partial_y u]dx\,dy$$

$$+\ u(x,y)\Big|_{(x,y)\in\partial D} = g(x,y)\ \ldots\ \begin{matrix}\bar{D} = D\cup\partial D \\ \partial D = \text{region boundary}\end{matrix}$$

Solution extremal have to be solution of Euler's equations and meet boundary conditions:

$$\frac{\partial F}{\partial u} - \partial_x\frac{\partial F}{\partial p} - \partial_y\frac{\partial F}{\partial q} = 0\ \ldots\ \begin{matrix}(x,y)\in\bar{D} \\ p(x,y) = \partial_x u,\ \ q(x,y) = \partial_y u\end{matrix}$$

Functional of higher order derivatives

$$J[y(x)] = \int_a^b F[x, y(x), y'(x),\ldots,y^{(n)}(x)]dx$$

$$+\ \ y^{(k)}(a) = A_k,\ \ y^{(k)}(b) = B_k\ \Big|_{k=0,1,\ldots,n-1}$$

Solution have to be solution of Euler's equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right] + \frac{d^2}{dx^2}\left[\frac{\partial F}{\partial y''}\right] - \ldots + (-1)^n\frac{d^n}{dx^n}\left[\frac{\partial F}{\partial y^{(n)}}\right] = 0$$